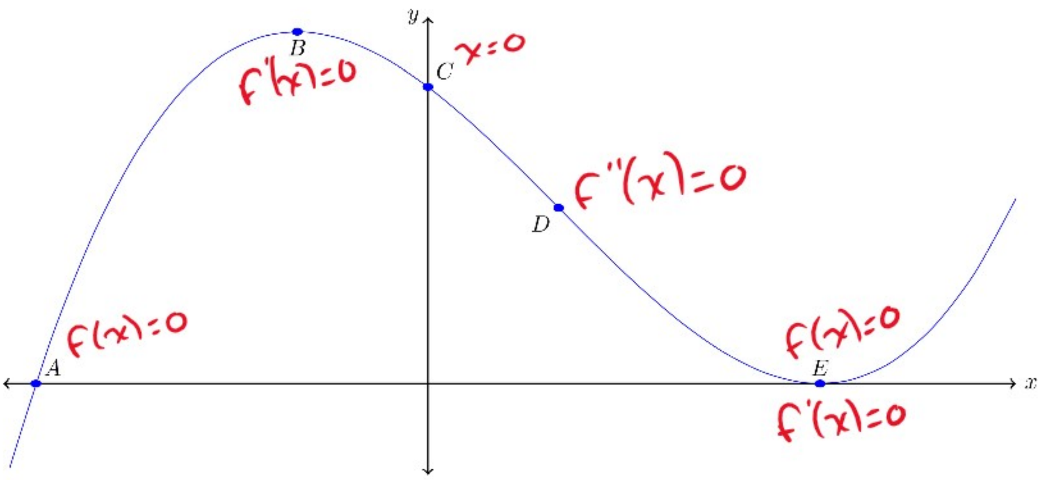
Name: Solutions

This assignment is **optional**. If you complete this assignment, the score you earn will replace your current lowest homework grade. There is **no late deadline** for this assignment and so it **must** be handed in on **Wednesday**. This assignment consists of nine questions, for a total of 35 points. To receive full credit you must **show all necessary work**. You should write your answers in the spaces provided, but if you require more space please *staple any extra sheets* you use to this assignment. If you are having trouble with any of the problems, look at the lecture notes and exercises in the lecture notes for help.



1. The graph above is of the function $f(x) = x^3 - 6x^2 - 36x + 216$. Use your knowledge of functions, the derivative and the second derivative to find the coordinates of each of the labelled points.

Point	A	B	C	D	E
Coordinates (x, y)	$(-6, 0)$	$(-2, 256)$	$(0, 216)$	$(2, 128)$	$(6, 0)$

$f(x) = x^3 - 6x^2 - 36x + 216 = x^2(x-6) - 36(x-6) = (x^2-36)(x-6) = (x+6)(x-6)(x-6)$

$f(x) = 0 \Rightarrow x = \pm 6$

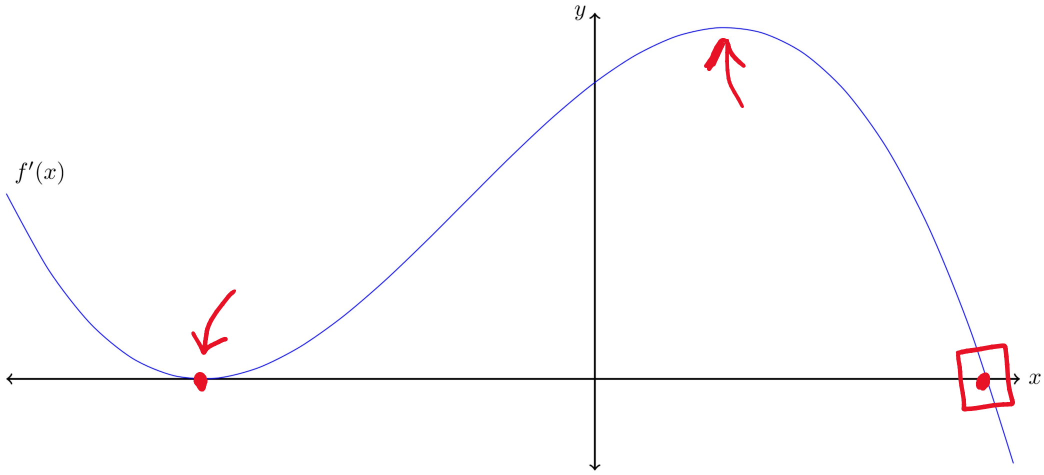
$f'(x) = 3x^2 - 12x - 36 = 3(x^2 - 4x - 12) = 3(x+2)(x-6)$

$f'(x) = 0 \Rightarrow x = -2, 6$

$f''(x) = 6x - 12 = 6(x-2)$

$f''(x) = 0 \Rightarrow x = 2$

(Hint: It will help to classify the points first in terms of their significance to the graph. Are they roots? Are they critical points? Knowing this you can set up equations to solve for the x-coordinates and then use the expression of $f(x)$ to find the y coordinates.)



2. The graph of $f'(x)$ is given above.
   (a) Does $f(x)$ have any critical points? If any exist draw a point on the graph at each of them.
   (b) Does $f(x)$ have any local minimums? If any exist, draw a circle around each of them on the graph.
   (c) Does $f(x)$ have any local maximums? If any exist, draw a box around each of them on the graph.
   (d) Does $f(x)$ have any points of inflection? If any exist, indicate them on the graph with an arrow.

   (Hint: Recall the definition of a critical point (stationary point) and a point of inflection. Also recall the second derivative test: Suppose $f'(a) = 0$.

   If $f''(a) > 0 \iff a$ is a local minimum
   If $f''(a) < 0 \iff a$ is a local maximum
   If $f''(a) = 0 \iff a$ we don't know.

   Note that if $f''(a) = 0$, then a is a point of inflection, but we cannot comment on its maximum/minimum-ness.)

3. Find the derivative of $y = \dfrac{(x^2+1)^3}{x^3+2x-1}$ $= (x^2+1)^3(x^3+2x-1)^{-1}$ <u>Product Rule</u>

$u(x) = (x^2+1)^3$

<u>Chain Rule</u>

$a(x) = x^3 \quad b(x) = x^2+1$

$u(x) = a(b(x))$

$a'(x) = 3x^2 \quad b'(x) = 2x$

$u'(x) = b'(x)a'(b(x))$
$= 2x \cdot 3(x^2+1)^2$
$= 6x(x^2+1)^2$

$v(x) = (x^3+2x-1)^{-1}$

<u>Chain Rule</u>

$f(x) = x^{-1} \quad g(x) = x^3+2x-1$

$v(x) = f(g(x))$

$f'(x) = -x^{-2} \quad g'(x) = 3x^2+2$

$v'(x) = g'(x)f'(g(x))$
$= (3x^2+2)(-(x^3+2x-1)^{-2})$
$= -(3x^2+2)(x^3+2x-1)^{-2}$

$y = u(x)v(x)$
$y' = u'(x)v(x) + u(x)v'(x)$
$= 6x(x^2+1)^2(x^3+2x-1)^{-1}$
$- (x^2+1)^3(3x^2+2)(x^3+2x-1)^{-2}$

Answer: $y' = 6x(x^2+1)^2(x^3+2x-1)^{-1} - (x^2+1)^3(3x^2+2)(x^3+2x-1)^{-2}$

Cont.

4. Let $C(q)$, $R(q)$ and $\pi(q)$ represent the cost, revenue and profit, in dollars, of producing q items.

   (a) If $C'(50) = 74$ and $R'(50) = 85$, approximately how much profit is earned by the 51st item?

   $\pi'(50) = R'(50) - C'(50) = 85 - 74 = 11$

   Answer: \$11

   (b) If $C'(90) = 68$ and $R'(90) = 62$, approximately how much profit is earned by the 91st item?

   $\pi'(90) = R'(90) - C'(90) = 62 - 68 = -6$

   Answer: $-\$6$

   (c) If $\pi(q)$ is a maximum when $q = 78$. Then which of the following is true: (circle the correct answer)

   $C'(78) < R'(78)$ $\quad$ $C'(78) = R'(78)$ $\quad$ $C'(78) > R'(78)$

   (Once you've answered this, make sure you understand why this is the case)

5. The demand equation for a product is $p = 29 - 0.01q$.

   (a) Find an expression for the revenue function $R(q)$.
   (Hint: Recall from Section 1.3 that the demand equation gives the price p when q items are being demanded. Recall also that revenue=price×quantity.)

   $R(q) = pq = (29 - 0.01q)q = 29q - 0.01q^2$

   Answer: $R(q) = 29q - 0.01q^2$

   (b) What quantity of this item should be produced to maximise revenue?

   $R'(q) = 29 - 0.02q = 0$
   $\Rightarrow 29 = 0.02q$
   $\Rightarrow 1450 = q$

   Answer: $q = 1450$

   (c) What is the price of each item at the quantity found in part (b)?

   $p = 29 - 0.01(1450) = 14.5$

   Answer: \$14.50

   (d) What is the total revenue received at the quantity found in part (b)?

   $R(1450) = 29(1450) - 0.01(1450)^2 = 21025$

   Answer: \$21025

Cont.

6. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given, in mg, by $q(t) = 40(e^{-t} - e^{-2t})$.

   (a) How much of the drug is in the bloodstream at time $t = 0$?

   $q(0) = 40(e^0 - e^0) = 40(1-1) = 0$

   Answer: 0mg

   (b) At what time is the quantity of drug in the bloodstream the highest?

   $q'(t) = 40(-e^{-t} + 2e^{-2t}) = 0$

   $\Longrightarrow -e^{-t} + 2e^{-2t} = 0$

   $\Longrightarrow 2e^{-2t} = e^{-t}$

   $\Longrightarrow 2e^{-t} = 1$

   $\Longrightarrow e^{-t} = 1/2$

   $\Longrightarrow -t = \ln(1/2)$

   $\Longrightarrow t = -\ln(1/2)$

   $\Longrightarrow t = \ln(2)$

   Answer: $t = \ln(2)$ hours

   (c) What is the maximum quantity of drug in the bloodstream?

   $q(\ln(2)) = 40(e^{-\ln(2)} - e^{-2\ln(2)}) = 40(\frac{1}{2} - \frac{1}{4}) = 40(\frac{1}{4}) = 10$

   Answer: 10mg

7. The cost and revenue functions of a company are $C(q) = 0.01q^3 - 0.75q^2 + 34q + 52$ and $R(q) = 26q$, where q is the number of items manufactured. To increase profits, should the company increase or decrease production from each of the following levels?
   (Hint: Remember, companies want to maximise profits.)

   (a) 5 items.

   $\pi(q) = R(q) - C(q) = -0.01q^3 + 0.75q^2 - 8q - 52$

   $\pi'(q) = -0.03q^2 + 1.5q - 8$

   $\pi''(q) = -0.06q + 1.5$

   $\pi'(5) = -0.03(5)^2 + 1.5(5) - 8$

   $= -1.25 < 0$

   $\pi''(5) = -0.06(5) + 1.5 = 1.2$

   Answer: Increase

   (b) 25 items.

   $\pi'(25) = -0.03(25)^2 + 1.5(25) - 8 = 10.75 > 0$

   Answer: Increase

   (c) 45 items.

   $\pi'(45) = -0.03(45)^2 + 1.5(45) - 8 = -1.25 < 0$

   $\pi''(45) = -0.06(45) + 1.5 = -1.2 < 0$

   Answer: Decrease

8. At a price of \$10 per ticket, a musical theatre group can fill every seat in the theatre, which has a capacity of 1300. For every additional dollar charged, the number of people buying tickets decreases by 65. At what price should the group sell each ticket to maximise revenue?

(Hint: First set up a function that relates price to quantity. Then proceed as in question 5.)

$q = 1300 - 65(p-10)$ Since # tickets decreases by 65 for each \$ above 10

$= 1300 + 650 - 65p = 1950 - 65p$

$\Rightarrow 65p = 1950 - q \Rightarrow p = \frac{1}{65}(1950 - q)$

$R(q) = pq = \frac{1}{65}(1950 - q)q = \frac{1}{65}(1950q - q^2)$

$R'(q) = \frac{1}{65}(1950 - 2q) = 0 \Rightarrow 1950 - 2q = 0 \Rightarrow 1950 = 2q$

$\Rightarrow q = 975$

$p = \frac{1}{65}(1950 - 975) = \frac{1}{65}(975) = 15$

Answer: \$15

9. The product of two positive numbers is 961. What is the minimum value of their sum?

(Hint: Denote the two numbers by x and y. So then $xy = 961$. Use this to write y in terms of x and create a function $S(x)$ to represent the sum of x and y in terms of x. Then use derivatives to find the minimum value.)

$xy = 961 \Rightarrow y = \frac{961}{x}$

$S(x) = x + y = x + \frac{961}{x}$

$S'(x) = 1 - 961x^{-2} = 0 \Rightarrow 1 = 961x^{-2}$

$\Rightarrow \frac{1}{961} = x^{-2} \Rightarrow 961 = x^2$

+ve since x,y are positive $\Rightarrow +\sqrt{961} = x = 31 \Rightarrow S(31) = 31 + \frac{961}{31} = 62$

Answer: 62

The End.